

KRONECKER LIMIT FORMULAS AND SCATTERING CONSTANTS FOR FERMAT CURVES

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ABSTRACT. Eisenstein series are real analytic functions which play a central role in spectral theory of the hyperbolic Laplacian. Kronecker limit formulas determine their connection to modular forms.

The main result of this work is Theorem 7.2 in which a Kronecker limit formula for a family of non-congruence subgroups associated with the Fermat curves is presented. As an application we can determine the scattering constants for the Fermat curves in Theorem 8.1.

1. INTRODUCTION

Eisenstein series are real analytic functions which play a central role in spectral theory of the hyperbolic Laplacian. They are defined via summing over a cusp of a subgroup of $\Gamma(1)$ (see e.g. formula (2.4.2)). Kronecker limit formulas show that these functions have a strong relation to modular forms. The classical Kronecker limit formula for $\Gamma(1)$ is

$$(1.0.1) \quad 4\pi \lim_{s \rightarrow 1} \left(E^{\Gamma(1)}(z, s) - \frac{3/\pi}{s-1} \right) = -\log \|\Delta(z)\|^2 + 24 \left(\frac{\zeta'(-1)}{\zeta(-1)} - \log(4\pi) + 1 \right),$$

(calculation similar to [Za]) where $E^{\Gamma(1)}(z, s)$ is the Eisenstein series, $\Delta(z)$ the well known Delta function, a modular form for $\Gamma(1)$, and $\|\cdot\|^2$ the Petersson norm that will be introduced in Definition 3.3.

A similar identity holds for subgroups of $\Gamma(1)$. Aim of this article is to establish such a formula for the groups Γ_N that are associated with the Fermat curves. The groups Γ_N are of particular interest, because they are, in most cases (in all but 4), non-congruence subgroups. Since non-congruence subgroups are normally much harder to handle, not much is known about them. The Fermat curves are an exceptional case due to their regularities and symmetries. We can work with them because of their nice description (see lemmas 5.3 and 5.5) and, in particular, because modular forms for Fermat curves were treated.

The main result of this article is Theorem 7.2 in which a Kronecker limit formula for the Fermat curves is presented. As an application we can determine the scattering constants for the Fermat curves in Theorem 8.1.

2. EISENSTEIN SERIES

Lemma 2.1. *We denote by $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ the upper half plane. The group*

$$\Gamma(1) := SL_2(\mathbb{Z}) / \{\pm 1\}$$

acts on \mathbb{H} via fractional linear transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az + b}{cz + d} \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \text{ and } z \in \mathbb{H}.$$

We can add a boundary to \mathbb{H} by joining the upper half plane with $\mathbb{P}^1(\mathbb{Q}) \cong \mathbb{Q} \cup \infty$, the rational projective line, and get $\bar{\mathbb{H}} := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$. The action of $\Gamma(1)$ on \mathbb{H} can be extended to $\bar{\mathbb{H}}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} ((p : q)) = (ap + bq : cp + dq) \quad \text{for } (p : q) \in \mathbb{P}^1(\mathbb{Q}).$$

Proof: See [Mi]. \square

At first, fix some notations. Let $\Gamma \subset \Gamma(1)$ be a subgroup. The classes of $\mathbb{P}^1(\mathbb{Q})$ with respect to the action of Γ are called cusps of Γ . We will use the word cusp for a representative of a cusp as well.

Let $\Gamma \subset \Gamma(1)$ be a finite index subgroup. For $S_j \in \mathbb{P}^1(\mathbb{Q})$ we will denote by γ_j a matrix $\gamma_j \in \Gamma(1)$ with $\gamma_j(\infty) = S_j$. Such a γ_j always exists. Furthermore, we normalize γ_j to

$$\sigma_j := \gamma_j \cdot \begin{pmatrix} \sqrt{b_j} & 0 \\ 0 & 1/\sqrt{b_j} \end{pmatrix},$$

with $b_j \in \mathbb{N}$ such that $\sigma_j^{-1}\Gamma_j\sigma_j = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle$, where $\Gamma_j := \text{Stab}_\Gamma(S_j)$. We call b_j the width of the cusp S_j .

For subgroups $\Gamma \subset \Gamma' \subset \Gamma(1)$ a cusp S'_k of Γ' decomposes into several cusps $\{S_j\}_{j \in J_k}$ of Γ and it holds $\bigcup_{j \in J_k} S_j = S'_k$. The cusps S_j are the subcusps of S'_k in Γ .

Definition 2.2. Let $\Gamma \subset \Gamma(1)$ be a finite index subgroup. For each cusp S_j there is a non-holomorphic Eisenstein series $E_j^\Gamma(z, s)$, which for $z \in \mathbb{H}$, $s \in \mathbb{C}$ and $\text{Re}(s) > 1$ is defined by the convergent series

$$(2.2.1) \quad E_j^\Gamma(z, s) = \sum_{\sigma \in \Gamma_j \backslash \Gamma} \text{Im}(\sigma_j^{-1}\sigma(z))^s.$$

We state some properties of Eisenstein series.

Proposition 2.3. The function $E_j^\Gamma(z, s)$ has a meromorphic continuation to the whole s -plane, with a simple pole in $s = 1$ of residue $3/(\pi \cdot [\Gamma(1) : \Gamma]) = 1/(\text{vol}(\Gamma))$.

Eisenstein series are automorphic forms: For all $\gamma \in \Gamma$ we have $E_j^\Gamma(\gamma(z), s) = E_j^\Gamma(z, s)$. They are eigenforms for the hyperbolic Laplacian Δ :

$$\Delta E_j^\Gamma(z, s) = s(s-1)E_j^\Gamma(z, s).$$

Proof: See [Ku] and [Iw2]. \square

Proposition 2.4. Let $\Gamma \subset \Gamma' \subset \Gamma(1)$ be finite index subgroups. Let S'_k be a cusp of Γ' and $\{S_j\}_{j \in J_k}$ the subcusps of S'_k in Γ . The widths will be denoted by w_k and b_j , respectively. Then we have the following relation for Eisenstein series

$$(2.4.1) \quad \sum_{j \in J_k} b_j^s E_j^\Gamma(z, s) = w_k^s E_k^{\Gamma'}(z, s).$$

Proof: Realizing that

$$(2.4.2) \quad E_j^\Gamma(z, s) = b_j^{-s} \sum_{\frac{d}{b_j} \in S_j} \frac{\text{Im}(z)^s}{|cz + d|^{2s}},$$

the statement follows by an easy calculation. \square

Eisenstein series admit a Fourier expansion. There are different normalizations that we can use in the expansion. The one below is the most useful for our purpose.

Proposition 2.5. Eisenstein series admit a Fourier expansion. The Fourier expansion of $E_j^\Gamma(z, s)$ at the cusp S_k is given by

$$(2.5.1) \quad \begin{aligned} E_j^\Gamma(\gamma_k(z), s) &= \delta_{jk} \frac{y^s}{b_j^s} + \pi^{1/2} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{1}{b_j^s b_k} \varphi_{jk,0}^\Gamma(s) y^{1-s} \\ &+ \sum_{m \neq 0} \frac{1}{b_j^s b_k} \varphi_{jk,m}^\Gamma(s) 2\pi^s \left| \frac{m}{b_k} \right|^{s-1/2} \Gamma(s)^{-1} y^{1/2} K_{s-1/2}(2\pi|m|y/b_k) e^{2\pi i m x/b_k}, \end{aligned}$$

where $z = x + iy$, $\Gamma(\cdot)$ is the Gamma function, $K_*(\cdot)$ the modified Bessel function and

$$\varphi_{jk,m}^\Gamma(s) := \sum_{c>0} \frac{1}{c^{2s}} \sum_{\substack{d \bmod b_k c \\ \exists \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \gamma_j^{-1} \Gamma \gamma_k}} e^{2\pi i m \frac{d}{b_k c}}.$$

Proof: Similar to T. Kubota [Ku]. The matrix σ_k in Kubota's proof has to be replaced by γ_k . \square

Remark 2.6. The expansion in Proposition 2.5 refers to the natural cusp width by using γ_k to move the cusp.

A normalization using the matrix σ_k (as it is done in [Ku]) would lead to a different expansion, which is obtained from Equation (2.5.1) by replacing z with $b_k z$. We will call this modified expansion the normalized expansion whereas the expansion from Proposition 2.5 is called the natural one.

We will give the following definitions using the normalized expansion. This has the advantage of coinciding with the ones in the literature (e.g. [Ku]) and of being symmetric.

Definition 2.7. For $\Gamma \subset \Gamma(1)$ a subgroup of finite index we define the scattering matrix (for the normalized Fourier expansion) to be

$$\Phi_\Gamma(s) := \left(\pi^{1/2} \frac{\Gamma(s-1/2)}{\Gamma(s)} \cdot \frac{1}{(b_j b_k)^s} \varphi_{jk,0}^\Gamma \right)_{j,k},$$

where j and k run over all cusps of Γ .

For all pairs j, k we define the (normalized) scattering constant C_{jk}^Γ to be the constant term at $s = 1$ of the Dirichlet series $(\Phi_\Gamma)_{jk}(s)$:

$$(2.7.1) \quad C_{jk}^\Gamma := \lim_{s \rightarrow 1} \left(\Phi_\Gamma(s)_{j,k} - \frac{1}{\text{vol}(\Gamma)(s-1)} \right).$$

Remark 2.8. If we take the natural Fourier expansion, see Remark 2.6, as basis to define the (natural) scattering matrix and the (natural) scattering constants, they change slightly:

We get $\frac{1}{b_j^s b_k}$ instead of $\frac{1}{(b_j b_k)^s}$ in the scattering matrix. The residue does not change but the scattering constants. Luckily, the difference is manageable and we have

$$(2.8.1) \quad \tilde{C}_{jk}^\Gamma = C_{jk}^\Gamma + \frac{\log(b_k)}{\text{vol}(\Gamma)},$$

where \tilde{C}_{jk}^Γ denotes the scattering constant coming from the constant term in the natural Fourier expansion.

Later on, for the Kronecker limit formulas, we will need another expansion. From Proposition 2.5 follows

Corollary 2.9. Let $\Gamma \subset \Gamma(1)$ be a finite index subgroup, S_j and S_k cusps of Γ . We have the following Fourier expansion

$$(2.9.1) \quad \lim_{s \rightarrow 1} \left(E_j^\Gamma(\gamma_k z, s) - \frac{1}{\text{vol}(\Gamma)(s-1)} \right) = \delta_{jk} \frac{y}{b_j} + \tilde{C}_{jk}^\Gamma - \frac{12 \log(y)}{[\Gamma(1) : \Gamma]} + \sum_{m \neq 0} \frac{1}{b_j b_k} \varphi_{jk,m}^\Gamma(1) e^{-2\pi |m|y/b_k} \cdot e^{2\pi m x/b_k}.$$

For the Fourier expansions we can get a relation analogous to Equation (2.4.1). Here, the calculations are more difficult and therefore we start with the preparatory

Lemma 2.10. *Let $\Gamma \subset \Gamma' \subset \Gamma(1)$ be finite index subgroups, S_j represent a cusp of Γ as well as one of Γ' , S'_k be a cusp of Γ' and $\{S_i\}_{i \in I_k}$ the subcusps of Γ such that $\cup_{i \in I_k} S_i = S'_k$. By b_* we denote the widths in Γ and by w_* the ones in Γ' , respectively. For $c \in \mathbb{N}$ and an integer m hold for the finite sums in the Fourier expansion of the Eisenstein series*

$$(2.10.1) \quad \frac{1}{b_j} \sum_{l \in I_k} \sum_{\substack{d \pmod{b_l c} \\ \exists \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \gamma_j^{-1} \Gamma \gamma_l}} e^{2\pi i m \frac{b_l}{w_k} \cdot \frac{d}{b_l c}} = \frac{1}{w_j} \sum_{\substack{d \pmod{w_k c} \\ \exists \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \gamma_j^{-1} \Gamma' \gamma_k}} e^{2\pi i m \frac{d}{w_k c}}.$$

Proof: In [Ku] it is shown that the scattering matrix is symmetric. From that, Proposition 2.4 and the Fourier expansion of Eisenstein series follows that we have the desired identity for $m = 0$:

$$\frac{1}{b_j} \sum_{l \in I_k} \sum_{\substack{d \pmod{b_l c} \\ \exists \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \gamma_j^{-1} \Gamma \gamma_l}} 1 = \frac{1}{b_j} \sum_{l \in I_k} \sum_{\substack{d \pmod{b_j c} \\ \exists \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \gamma_l^{-1} \Gamma \gamma_j}} 1 = \frac{1}{w_j} \sum_{\substack{d \pmod{w_j c} \\ \exists \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \gamma_k^{-1} \Gamma' \gamma_j}} 1 = \frac{1}{w_j} \sum_{\substack{d \pmod{w_k c} \\ \exists \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \gamma_j^{-1} \Gamma' \gamma_k}} 1$$

That gives us the number of summands in the double sum on the left in Equation (2.10.1) comparative to the sum on the right: (sums left) = $\frac{b_j}{w_j} \cdot$ (sum right).

There is a well known decomposition of $\gamma_j^{-1} \Gamma \gamma_k$ into double cosets (see [Iw1] slightly modified):

$$(2.10.2) \quad \gamma_j^{-1} \Gamma \gamma_k = \delta_{jk} \left\langle \begin{pmatrix} 1 & b_j \\ 0 & 1 \end{pmatrix} \right\rangle \cup \bigcup_{c \geq 0} \bigcup_{d \pmod{b_k c}} \left\langle \begin{pmatrix} 1 & b_j \\ 0 & 1 \end{pmatrix} \right\rangle \begin{pmatrix} * & * \\ c & d \end{pmatrix} \left\langle \begin{pmatrix} 1 & b_k \\ 0 & 1 \end{pmatrix} \right\rangle,$$

where the union is taken over all pairs c, d such that there is $\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \gamma_j^{-1} \Gamma \gamma_k$. Since each d in the decomposition determines the upper left entry of the matrix $a \pmod{b_j c}$, we get

$$(2.10.3) \quad \gamma_j^{-1} \Gamma \gamma_k = \delta_{jk} \left\langle \begin{pmatrix} 1 & b_j \\ 0 & 1 \end{pmatrix} \right\rangle \cup \bigcup_{c \geq 0} \bigcup_{a \pmod{b_j c}} \left\langle \begin{pmatrix} 1 & b_j \\ 0 & 1 \end{pmatrix} \right\rangle \begin{pmatrix} a & * \\ c & * \end{pmatrix} \left\langle \begin{pmatrix} 1 & b_k \\ 0 & 1 \end{pmatrix} \right\rangle.$$

Hence, the d 's on the left hand side of Equation (2.10.1) correspond to a 's in matrices $\begin{pmatrix} a & * \\ c & d \end{pmatrix}$. With the decomposition (2.10.3), one can show that all a 's in the matrices are distinct.

From $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \gamma_l^{-1} \Gamma \gamma_j$ follows $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \gamma_k^{-1} \Gamma' \gamma_j$. Thus, the reduction of $a \pmod{w_j}$ creates a class in the double coset decomposition of Γ' . Hence, all a 's lie in

$$A = \left\{ a' + n w_j \mid 0 \leq a' < w_j c \text{ such that } \exists \begin{pmatrix} a' & * \\ c & * \end{pmatrix} \in \gamma_j^{-1} \Gamma' \gamma_k, 0 \leq n < \frac{b_j}{w_j} \right\}.$$

Since $|A|$ is just the required number, the whole of A is the set of a 's. Now we see that the reduction modulo w_j of $\frac{b_j}{w_j}$ different a 's coincide, the reduction of the corresponding d 's likewise. Hence, the statement follows. \square

Now, we can prove

Proposition 2.11. *Let $\Gamma \subset \Gamma' \subset \Gamma(1)$ be finite index subgroups. Let S_j and S_k be cusps of cusp width b_* (in Γ) and w_* (in Γ'), respectively. Then we have*

$$(2.11.1) \quad \frac{1}{b_j^{1-s}} \sum_{\gamma \in \Gamma \backslash \Gamma'} E_j^\Gamma(\gamma \gamma_k z, s) = \frac{1}{w_j^{1-s}} E_j^{\Gamma'}(\gamma_k z, s).$$

Proof: To understand what happens in the sum, we choose a suitable system of representatives. We have

$$\Gamma' = \bigcup_{S_l \sim_{\Gamma'} S_k} \bigcup_{0 \leq n < \frac{b_l}{w_k}} \Gamma \gamma_{lk} \tau_{k, w_k n},$$

where the first union is taken over representatives for all cusps S_l of Γ that are Γ' -equivalent to S_k . The matrices that occur are all from $\Gamma(1)$ with $\gamma_{lk}(S_k) = S_l$, $\tau_{k, w_k n} = \gamma_k \tau_{w_k n} \gamma_k^{-1} \in \text{Stab}_{\Gamma'}(S_k)$

and $\tau_{w_k n} = \begin{pmatrix} 1 & w_k n \\ 0 & 1 \end{pmatrix}$. Then

$$\begin{aligned} \sum_{\gamma \in \Gamma \backslash \Gamma'} E_j^\Gamma(\gamma \gamma_k z, s) &= \sum_{S_l \sim S_k} \sum_{0 \leq n < \frac{b_l}{w_k}} E_j^\Gamma(\gamma_{lk} \gamma_k \tau_{w_k n} \gamma_k^{-1} \gamma_k z, s) \\ &= \sum_{S_l \sim S_k} \sum_{0 \leq n < \frac{b_l}{w_k}} E_j^\Gamma(\gamma_l(z + w_k n), s) \end{aligned}$$

with $\gamma_{lk} \gamma_k = \gamma_l$ which fulfills $\gamma_l(\infty) = S_l$.

Now, we look at the sum of the Fourier expansions (use Equation (2.5.1)) and we get

$$\begin{aligned} \sum_{\gamma \in \Gamma \backslash \Gamma'} E_j^\Gamma(\gamma \gamma_k z, s) &= \delta_{jk} \frac{b_j}{b_j^s w_j} y^s + \pi^{1/2} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{b_j}{b_j^s w_j w_k} \left(\sum_{c>0} \frac{1}{c^{2s}} \sum_{S_l \sim S_k} \sum_{\substack{d \bmod b_l c \\ \exists \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \gamma_j^{-1} \Gamma \gamma_l}} 1 \right) y^{1-s} \\ &\quad + \sum_{m \neq 0} \sum_{S_l \sim S_k} \frac{1}{b_j^s b_l} \left(\sum_{c>0} \frac{1}{c^{2s}} \sum_{\substack{d \bmod b_l c \\ \exists \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \gamma_j^{-1} \Gamma \gamma_l}} e^{2\pi i m \frac{d}{b_l c}} \right) 2\pi^s \left| \frac{m}{b_l} \right|^{s-1/2} \\ &\quad \cdot \Gamma(s)^{-1} y^{1/2} K_{s-1/2}(2\pi |m| y / b_l) \left(\sum_{0 \leq n < \frac{b_l}{w_k}} e^{2\pi i n / (b_l / w_k)} \right) e^{2\pi i m x / b_l}. \end{aligned}$$

On the constant term we can apply Lemma 2.10. For the higher terms we first conclude with

$$\sum_{0 \leq n < k} e^{2\pi i m \frac{n}{k}} = \begin{cases} k & \text{if } k|m \\ 0 & \text{elsewise,} \end{cases}$$

that for many m the coefficient is zero and apply Lemma 2.10 on the remaining ones. If we compare the result with the Fourier expansion of $E_j^{\Gamma'}(\gamma_k z, s)$ we get the statement. \square

Similarly, an identity holds for the values in $s = 1$.

Corollary 2.12. *With the notations from Proposition 2.11 we have*

$$(2.12.1) \quad \sum_{\gamma \in \Gamma \backslash \Gamma'} \lim_{s \rightarrow 1} \left(E_j^\Gamma(\gamma \gamma_k z, s) - \frac{1}{\text{vol}(\Gamma)(s-1)} \right) = \lim_{s \rightarrow 1} \left(E_j^{\Gamma'}(\gamma_k z, s) - \frac{1}{\text{vol}(\Gamma')(s-1)} \right) - \frac{1}{\text{vol}(\Gamma')} \cdot \log \left(\frac{b_j}{w_j} \right).$$

3. KRONECKER LIMIT FORMULAS

To establish Kronecker limit formulas we need modular forms. Here, we just give the definitions needed. An introduction to modular forms can be found in [Mi].

Definition 3.1. *We define an action of $\Gamma(1)$ on functions $f : \mathbb{H} \rightarrow \mathbb{C}$ via*

$$f|_k \gamma(z) = (cz + d)^{-k} f(\gamma z),$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ and $k \in \mathbb{Z}$. This action is called the slash operator of weight k or the k -th slash operator.

Let $\Gamma \subset \Gamma(1)$ be a subgroup of finite index, k an integer. A meromorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ behaves automorphically of weight k with respect to Γ if

$$f|_k \gamma(z) = f(z) \quad \forall \gamma \in \Gamma.$$

Then $f|_k\gamma_j(z)$ is b_j -periodic, i.e. $f|_k\gamma_j(z + b_j) = f|_k\gamma_j(z)$, where b_j is the cusp width of S_j . Therefore there exists a function g on $D \setminus \{0\}$ (the punctured unit disc) such that

$$f|_k\gamma_j(z) = g(e^{2\pi iz}) \quad z \in \mathbb{H}.$$

The function g is meromorphic on $D \setminus \{0\}$, since f is meromorphic. We say that f is meromorphic, is holomorphic in the cusp S_j if g extends meromorphically, holomorphically to 0, respectively.

Definition 3.2. A holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ is called modular function with respect to Γ if it behaves automorphically of weight 0 and is meromorphic in all cusps of Γ ; a holomorphic function $f(z)$ is called a modular form (of weight k with respect to Γ) if it behaves automorphically of weight k and is holomorphic in all cusps.

The set of modular forms of weight k with respect to Γ is denoted by $M_k(\Gamma)$, it generates a ring graded by the weight.

Definition 3.3. Let $f(z) \in M_k(\Gamma)$ be a modular form for a finite index subgroup $\Gamma \subset \Gamma(1)$. Then we define its Petersson norm via

$$(3.3.1) \quad \|f(z)\|^2 := |f(z)|^2 \operatorname{Im}(z)^k.$$

Now, we can formulate the Kronecker limit formula:

Proposition 3.4. Let $\Gamma \subset \Gamma(1)$ be a subgroup, S_j a cusp of Γ . Suppose there is a modular form $f_j^\Gamma \in M_k(\Gamma)$ ($k \in \mathbb{N}$), that only vanishes in the cusp S_j . Then there is a constant $A \in \mathbb{R}$ (depending on f_j^Γ) such that

$$(3.4.1) \quad 4\pi \lim_{s \rightarrow 1} \left(E_j^\Gamma(z, s) - \frac{1}{\operatorname{vol}(\Gamma)(s-1)} \right) = -\frac{1}{[\Gamma(1) : \Gamma] \cdot \frac{k}{12}} \log \|f_j^\Gamma\|^2 + A.$$

Proof: Examine the action of the hyperbolic Laplace operator $\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ on the functions $4\pi \lim_{s \rightarrow 1} \left(E_j^\Gamma(z, s) - \frac{1}{\operatorname{vol}(\Gamma)(s-1)} \right)$ and $-\log \|f_j^\Gamma\|^2$. We have for the left side of Equation (3.4.1) (for the expansion in a cusp S_l)

$$\begin{aligned} & \Delta \left(4\pi \lim_{s \rightarrow 1} \left(E_j^\Gamma(\gamma_l z, s) - \frac{1}{\operatorname{vol}(\Gamma)(s-1)} \right) \right) \\ &= \Delta \left(4\pi \left(\delta_{jl} \frac{y}{b_l} + \tilde{C}_{jl}^\Gamma - \frac{3 \log(y)}{\pi [\Gamma(1) : \Gamma]} + \sum_{n \neq 0} a_n(y, 1) q^{xn/b_l} \right) \right) \\ &= \frac{12}{[\Gamma(1) : \Gamma]}, \end{aligned}$$

with $q = e^{2\pi i}$ and $z = x + iy$, where the formula for the expansion can be found in Corollary 2.9. On the other side,

$$\begin{aligned} \Delta(-\log \|f_j^\Gamma\|^2) &= \Delta(-\log(|f_j^\Gamma|^2 y^k)) \\ &= -\Delta(\log(f_j^\Gamma)) - \Delta(\log(\overline{f_j^\Gamma})) - \Delta(\log(y^k)) \\ &= 0 + 0 + k. \end{aligned}$$

Hence:

$$\Delta \left(4\pi \lim_{s \rightarrow 1} \left(E_j^\Gamma(z, s) - \frac{1}{\operatorname{vol}(\Gamma)(s-1)} \right) + \frac{1}{[\Gamma(1) : \Gamma] \cdot \frac{k}{12}} \log \|f_j^\Gamma\|^2 \right) = 0$$

The spectral decomposition of the Laplacian had been studied, see [Iw2], for functions that are square integrable (the space $\mathfrak{L}(Y_\Gamma)$). To find out if we can use the result from [Iw2], we will study the behavior of $4\pi \lim_{s \rightarrow 1} \left(E_j^\Gamma(z, s) - \frac{1}{\operatorname{vol}(\Gamma)(s-1)} \right) + \frac{1}{[\Gamma(1) : \Gamma] \cdot \frac{k}{12}} \log \|f_j^\Gamma\|^2$ in the cusps. For that, we will compare the expansions. We have seen the expansion of the Eisenstein series in a cusp S_l in Corollary 2.9. The expansion $f_j^\Gamma|_{S_l}$ has the form $d_m q^{zm/b_l} (1 + \sum_n d_n q^{zn/b_l})$, with $q = e^{2\pi i}$, $m \in \{0, [\Gamma(1) : \Gamma] \cdot \frac{k}{12}\}$ and $m = 0$ if and only if $S_j \neq S_l$, since f_j^Γ only vanishes in the cusp S_j

and the vanishing order is $[\Gamma(1) : \Gamma] \cdot \frac{k}{12}$ (by the theory of modular forms). Therefore, we have $(z = x + iy)$

$$\begin{aligned}
 \log \|f_j|_{S_l}\|^2 &= \log(y^k |f_j|_{S_l}|^2) \\
 &= k \cdot \log(y) + 2 \operatorname{Re} \log(f_j|_{S_l}) \\
 &= k \cdot \log(y) + 2 \operatorname{Re} \log \left(d_m q^{zm/b_l} \left(1 + \sum_{n>0} d_n q^{zn/b_l} \right) \right) \\
 &= k \cdot \log(y) + 2 \operatorname{Re} \log(d_m) - \delta_{jl} 4\pi \frac{y}{b_l} \cdot [\Gamma(1) : \Gamma] \cdot \frac{k}{12} \\
 &\quad + 2 \operatorname{Re} \log \left(1 + \sum_{n>0} d_n q^{zn/b_l} \right) \\
 &= k \cdot \log(y) + 2 \operatorname{Re} \log(d_m) - \delta_{jl} 4\pi \frac{y}{b_l} \cdot [\Gamma(1) : \Gamma] \cdot \frac{k}{12} + 2 \operatorname{Re} \left(\sum_{n>0} \tilde{d}_n q^{zn/b_l} \right),
 \end{aligned}$$

where the \tilde{d}_n are suitable such that $\log(1 + \sum_{n>0} d_n q^{zn/b_l}) = \sum_{n>0} \tilde{d}_n q^{zn/b_l}$.

Now we can see that the value of

$$4\pi \lim_{s \rightarrow 1} \left(E_j^\Gamma(z, s) - \frac{1}{\operatorname{vol}(\Gamma)(s-1)} \right) + \frac{1}{[\Gamma(1) : \Gamma] \cdot \frac{k}{12}} \log \|f_j^\Gamma\|^2$$

in S_l is bounded:

$$\begin{aligned}
 &\lim_{z \rightarrow i\infty} \left(4\pi \lim_{s \rightarrow 1} \left(E_j^\Gamma(\gamma_l z, s) - \frac{1}{\operatorname{vol}(\Gamma)(s-1)} \right) + \frac{1}{[\Gamma(1) : \Gamma] \cdot \frac{k}{12}} \log \|f_j^\Gamma|_{S_l}\|^2 \right) \\
 &= \lim_{z \rightarrow i\infty} \left(4\pi \tilde{C}_{jl}^\Gamma + \frac{12 \log(4\pi)}{[\Gamma(1) : \Gamma]} + 2 \operatorname{Re} \log(d_m) + \sum_{n \neq 0} a_n e^{-2\pi|n|\frac{y}{b_l}} e^{2\pi i \frac{x}{b_l}} \right. \\
 &\quad \left. + 2 \operatorname{Re} \left(\sum_{n>0} \tilde{d}_n e^{2\pi i n \frac{z}{b_l}} \right) \right) \\
 &= 4\pi \tilde{C}_{jl}^\Gamma + \frac{12 \log(4\pi)}{[\Gamma(1) : \Gamma]} + 2 \operatorname{Re} \log(d_m),
 \end{aligned}$$

where S_l was chosen arbitrarily. Therefore, we have

$$4\pi \lim_{s \rightarrow 1} \left(E_j^\Gamma(z, s) - \frac{1}{\operatorname{vol}(\Gamma)(s-1)} \right) + \frac{1}{[\Gamma(1) : \Gamma] \cdot \frac{k}{12}} \log \|f_j\|^2 \in \mathfrak{L}(Y_\Gamma).$$

The spectral decomposition in [Iw2] shows that the kernel of the Laplacian are the constant functions. If we take a closer look at the functions involved here, we see that we get a real number. \square

Remark 3.5. The constant A from Proposition 3.4 in Formula (3.4.1) can be calculated by comparison of the Fourier expansions.

It is independent of the cusp S_l in which the expansion is taken: To explain this, we may start with the functions expanded in ∞ to see what happens if we pass on to another cusp. Changing from ∞ to the cusp S_l means changing z to $\gamma_l z$, where $\gamma_l \in \Gamma(1)$ with $\gamma_l^{-1} \Gamma_l \gamma_l = \langle \begin{pmatrix} 1 & b_l \\ 0 & 1 \end{pmatrix} \rangle$. Thus, only the parts of the function change that depend on z . These parts coincide for both sides of Equation 3.4.1 such that their difference stays the same.

4. KRONECKER LIMIT FORMULAS FOR $\Gamma(2)$

The group $\Gamma(2) \subset \Gamma(1)$ is a free subgroup of index 6 with two generators

$$(4.0.1) \quad \gamma_1 := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \gamma_2 := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

It has three cusps of width 2 that are (see [Sh]):

$$(4.0.2) \quad \begin{aligned} & \{(p : q) \in \mathbb{P}^1(\mathbb{Q}) \mid (p, q) = 1, (p, q) \equiv (0, 1) \pmod{2}\} \\ & \{(p : q) \in \mathbb{P}^1(\mathbb{Q}) \mid (p, q) = 1, (p, q) \equiv (1, 1) \pmod{2}\} \\ & \{(p : q) \in \mathbb{P}^1(\mathbb{Q}) \mid (p, q) = 1, (p, q) \equiv (1, 0) \pmod{2}\}, \end{aligned}$$

thus a system of representatives is $\{0, 1, \infty\}$.

We define (with $q = e^{2\pi iz}$):

$$(4.0.3) \quad \begin{aligned} \theta^2(z) &= \prod_{n \geq 1} (1 - q^n)^4 \left(1 + q^{n-1/2}\right)^8 \\ \lambda(z) &= -\frac{1}{16} q^{-1/2} \prod_{n \geq 1} \left(\frac{1 - q^{n-1/2}}{1 + q^n}\right)^8 \\ (1 - \lambda)(z) &= \frac{1}{16} q^{-1/2} \prod_{n \geq 1} \left(\frac{1 + q^{n-1/2}}{1 + q^n}\right)^8 \end{aligned}$$

Then $\theta^2(z)$ is a modular form for $\Gamma(2)$ of weight 2, the functions $\lambda(z)$ and $(1 - \lambda)(z)$ are modular functions for the same group. They have the following divisors

$$\operatorname{div} \theta^2 = 1 \cdot 1, \quad \operatorname{div} \lambda = 1 \cdot 0 - 1 \cdot \infty, \quad \operatorname{div}(1 - \lambda) = 1 \cdot 1 - 1 \cdot \infty.$$

(See [Ya], or, for more background information, [Mi] and [EMOT].)

Hence, the modular forms

$$(4.0.4) \quad G_0(z) := \frac{\lambda(z)}{1 - \lambda(z)} \theta^2(z)$$

$$(4.0.5) \quad G_1(z) := \theta^2(z)$$

$$(4.0.6) \quad G_\infty(z) := \frac{1}{1 - \lambda(z)} \theta^2(z)$$

from $M_2(\Gamma(2))$ have divisors

$$\operatorname{div} G_0 = 1 \cdot 0, \quad \operatorname{div} G_1 = 1 \cdot 1, \quad \operatorname{div} G_\infty = 1 \cdot \infty.$$

Proposition 4.1 (Kronecker limit formula for $\Gamma(2)$). *For the group $\Gamma(2)$ holds*

$$(4.1.1) \quad 4\pi \lim_{s \rightarrow 1} \left(E_j^{\Gamma(2)}(z, s) - \frac{1}{\operatorname{vol}(\Gamma(2))(s-1)} \right) = -\log \|G_j(z)\|^2 + 4 \left(\frac{\zeta'(-1)}{\zeta(-1)} - \log(4\pi) + 1 + \frac{1}{6} \log(2) \right),$$

where $j \in \{0, 1, \infty\}$ denotes one of the three cusps of $\Gamma(2)$, $E_j^{\Gamma(2)}(z, s)$ is an Eisenstein series and G_j the corresponding modular form from one of the equations (4.0.4) to (4.0.6).

Proof: We will compare the expansions of both functions involved in the cusp ∞ . In ∞ we get expansions (with $z = x + iy$)

$$4\pi \lim_{s \rightarrow 1} \left(E_j^{\Gamma(2)}(z, s) - \frac{1}{\operatorname{vol}(\Gamma(2))(s-1)} \right) = \sum_{m \in \mathbb{Z}} e_{j,m}(y) e^{\pi i m x}$$

and

$$-\log \|G_j(z)\|^2 = \sum_{m \in \mathbb{Z}} g_{j,m}(y) e^{\pi i m x}.$$

The identity $e_{j,m}(y) = g_{j,m}(y)$ for all $m \neq 0$ follows from Proposition 3.4. Therefore, we just have to deal with $m = 0$. The coefficient $g_{j,0}(y)$ can easily be derived from the product description in Equations (4.0.3) and the definition of the Petersson norm (3.3). In all three cusps S_j holds

$$(4.1.2) \quad g_{j,0}(y) = \delta_{j\infty} (2\pi y - 8 \log(2)) - 2 \log(y).$$

If we regard the Eisenstein series, we get from Equation (2.9.1) that

$$(4.1.3) \quad e_{j,0}(y) = \delta_{j\infty} 2\pi y + 4\pi \tilde{C}_{j\infty}^{\Gamma(2)} - 2 \log(y).$$

The scattering constants for the group $\Gamma(2)$ can be calculated with the description of the cusps (Equation (4.0.2)), either by using results by Huxley [Hu] or directly from the Fourier expansion (as it had been done in [Po2]): One gets

$$\begin{aligned} \tilde{C}_{jk}^{\Gamma(2)} &= \lim_{s \rightarrow 1} \left(\frac{1}{2 \cdot 2^s} \pi^{1/2} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{2^{2s} - 2}{2^{2s} - 1} \frac{\zeta(2s-1)}{\zeta(2s)} - \frac{1}{2\pi(s-1)} \right) \\ &= \frac{1}{\pi} \left(\frac{\zeta'(-1)}{\zeta(-1)} - \log(4\pi) + 1 + \frac{1}{6} \log(2) \right) \\ \tilde{C}_{jj}^{\Gamma(2)} &= \lim_{s \rightarrow 1} \left(\frac{1}{2^s} \pi^{1/2} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{1}{2^{2s} - 1} \frac{\zeta(2s-1)}{\zeta(2s)} - \frac{1}{2\pi(s-1)} \right) \\ &= \frac{1}{\pi} \left(\frac{\zeta'(-1)}{\zeta(-1)} - \log(4\pi) + 1 - \frac{11}{6} \log(2) \right), \end{aligned}$$

where the first formula holds when $S_j \neq S_k$ and the second one in case of equality. With this information we can compare equations (4.1.2) and (4.1.3) to obtain the statement. \square

5. BASICS ON FERMAT CURVES

As a projective curve the well known Fermat curve is given by

Definition 5.1. *Let $N \in \mathbb{N}$. The N -th Fermat curve is given by the equation*

$$(5.1.1) \quad F_N : \quad X^N + Y^N = Z^N.$$

Lemma 5.2. *Consider the map*

$$(5.2.1) \quad \begin{aligned} \beta_N : \quad F_N &\longrightarrow \mathbb{P}^1. \\ (X : Y : Z) &\longmapsto (X^N : Z^N) \end{aligned}$$

Its degree is N^2 . It is ramified only above the points $0, 1, \infty$ and the ramification points are

$$(5.2.2) \quad \begin{aligned} a_j &:= (0 : \zeta^j : 1) \\ b_j &:= (\zeta^j : 0 : 1) \\ c_j &:= (\epsilon \zeta^j : 1 : 0), \end{aligned}$$

where $\zeta = e^{2\pi i/N}$ is the first primitive N -th root of unity, $j \in \{0, \dots, N-1\}$ and $\epsilon = e^{\pi i/N}$. Each point has ramification index N .

Proof: Simple calculation. \square

There is a subgroup Γ_N of $\Gamma(2)$ given by the monodromy of the cover β_N with the property

$$\Gamma_N \setminus \mathbb{H} \cong F_N(\mathbb{C}) \setminus \{\text{ramification points of } \beta_N\}.$$

The group Γ_N can be described as in the following

Lemma 5.3. *The group Γ_N is the kernel of*

$$\begin{aligned} \Gamma(2) &\longrightarrow \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}, \\ \gamma &\longmapsto (R_1(\gamma), R_2(\gamma)) \pmod{N}, \end{aligned}$$

where $R_i(\gamma)$ denotes the number of generators γ_i ($i \in \{0, 1\}$) for $\Gamma(2)$ (see Equation (4.0.1)) that occur in the word description of γ :

Let $\gamma \in \Gamma(2)$ be given via its word in γ_1 and γ_2 as $\gamma = \prod_{i=1}^n \kappa_i^{r_i}$ with $n \in \mathbb{N}$, $r_i \in \mathbb{Z}$, $\kappa_i \in \{\gamma_1, \gamma_2\}$. Then

$$R_1(\gamma) = \sum_{\kappa_i=\gamma_1} r_i \quad \text{and} \quad R_2(\gamma) = \sum_{\kappa_i=\gamma_2} r_i.$$

Proof: See [MR]. □

Remark 5.4. The subgroup Γ_N from Lemma 5.3 is a non-congruence subgroup for all N but 1, 2, 4 and 8 (see [PS]).

Further facts about Γ_N .

Lemma 5.5. We have $\Gamma_N \triangleleft \Gamma(1)$, $[\Gamma(1) : \Gamma_N] = 6N^2$ and the group has $3N$ cusps, all of same width $b = 2N$. A system of representatives for the cosets $\Gamma_N \backslash \Gamma(2)$ is

$$(5.5.1) \quad \{\gamma_1^a \gamma_2^b\} \quad \text{with} \quad a, b \in 0, \dots, N-1.$$

A system of representatives for the cusps is $S = S_0 \cup S_1 \cup S_\infty$ with

$$(5.5.2) \quad S_0 = \{0, 2, \dots, 2N-2\}, S_1 = \{1, 3, \dots, 2N-1\}, S_\infty = \left\{ \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2N} \sim_{\Gamma_N} \infty \right\}.$$

The cusps in S_i are $\Gamma(2)$ -equivalent to i ($i \in \{0, 1, \infty\}$).

Proof: The fact that Γ_N is normal, the index, the number of cusps and the representatives for the cosets follow from the lemmas 5.2 and 5.3.

To prove that S contains exactly one representative for all cusps, it is enough to show that all elements of S are non-equivalent under Γ_N . Since cusps from different subsets S_0, S_1, S_∞ are non-equivalent under $\Gamma(2)$ (see Equation (4.0.2)), we have to compare cusps out of the same subset only. Hence, we have to express a general matrix that maps cusps out of one subset to each other in the generators γ_1, γ_2 of $\Gamma(2)$ and find out if they are in Γ_N .

This reduces the problem to the examination of the following words, where j and k denote cusps from S_i , $i \in \{0, 1, \infty\}$:

$$\gamma_{ik}^{-1} \kappa_i^m \gamma_{ij} \quad \text{with} \quad m \in \mathbb{Z}; \text{Stab}_{\Gamma(2)}(i) = \langle \kappa_i \rangle; \gamma_{ij}, \gamma_{ik} \in \Gamma(2) : \gamma_{ij}(j) = \gamma_{ik}(k) = i.$$

We have $\kappa_0 = \gamma_2, \kappa_1 = (\gamma_2 \gamma_1^{-1}), \kappa_\infty = \gamma_1$. In the cases $i = 0$ and $i = 1$ powers of γ_1 do for γ_{ki} as well as for γ_{ji} and for $i = \infty$ powers of γ_2 . Combined with the fact that the smallest value of $|m|$, for which γ_1^m or γ_2^m lie in Γ_N , is $|m| = N$, we get the result. □

There is a 1-1-correspondence between the cusps of Γ_N and the ramification points of β_N that can be made explicit.

Proposition 5.6. A possible identification of ramification points of the Belyi map β_N (Equation (5.2.2)) and the cusps of the group Γ_N (Equation (5.5.2)) is

$$\begin{aligned} (0 : \zeta^n : 1) &\longleftrightarrow 2N - 2n \\ (\zeta^n : 0 : 1) &\longleftrightarrow 2N - 2n - 1 \\ (\epsilon \zeta^n : 1 : 0) &\longleftrightarrow \frac{1}{2N - 2n} \end{aligned} \quad n \in \{0, \dots, N-1\}.$$

Proof: The situation is the following:

$$\begin{array}{ccc} F_N(\mathbb{C}) \setminus \{a_j, b_j, c_j\}_{j=0 \dots N-1} & \xrightarrow{\sim} & \Gamma_N \setminus \mathbb{H} \\ \downarrow & & \downarrow \\ \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} & \xrightarrow{\sim} & \Gamma(2) \setminus \mathbb{H} \end{array}$$

The lower isomorphism is given via $\lambda(x)$ (see equation (4.0.3); we compose it with an automorphism to fix 0, 1, ∞). The isomorphism preserves the orientation.

Via lifting N -fold circles around 0 and ∞ we get the orders of the cusps around $(0 : 1 : 1)$ and $(\epsilon : 1 : 0)$.

For $(0 : 1 : 1)$ take the path $\frac{1}{2}e^{2\pi i N \lambda}$ ($\lambda \in [0, 1]$) on \mathbb{P}^1 . One of its lifts on F_N under β_N meets all the preimages of the real line that connect the cusp $(0 : 1 : 1)$ with the $b_j's$ and $c_j's$ (Equation (5.2.2)) in the following order:

$$(1 : 0 : 1), (\epsilon : 1 : 0), (\zeta_N : 0 : 1), (\epsilon \zeta_N : 1 : 0), \dots, (\zeta_N^{N-1} : 0 : 1), (\epsilon \zeta_N^{N-1} : 1 : 0).$$

For ∞ we can get a corresponding result by lifting $2e^{2\pi i N \lambda}$:

$$(1 : 0 : 1), (0 : 1 : 1), (\zeta_N : 0 : 1), (0 : \zeta_N : 1), \dots, (\zeta_N^{N-1} : 0 : 1), (0 : \zeta_N^{N-1} : 1).$$

A lift of a circle around the cusp S_i on the F_N side corresponds to the application of a matrix κ_i , that generates $\text{Stab}_{\Gamma(2)}(S_i)$, on the side of $\Gamma_N \setminus \mathbb{H}$. The quotient $\Gamma_N \setminus \mathbb{H}$ is represented by a fundamental domain $\mathcal{F}_N \subset \mathbb{H}$ for Γ_N . It has a tessellation by fundamental domains of $\Gamma(2)$, that are triangles with vertices 0, 1 and ∞ . When κ_i acts on \mathcal{F}_N it interchanges the triangles around the cusp S_i and we get an order of the cusps as before.

In the case of 0 we have to take $\kappa_0 = \gamma_2^{-1}$ to describe a turn in positive direction. Then we get $\gamma_2^{-n}(\infty) = \frac{1}{-2n} \sim_{\Gamma_N} \frac{1}{2N-2n}$ (the equivalence is given via $\gamma_2^N \in \Gamma_N \quad \forall N$) as well as $\gamma_2^{-n}(1) = \frac{1}{-2n+1} \sim_{\Gamma_N} 2N-2n+1$ (via $\gamma_1^{N+n-1}(\gamma_2 \gamma_1^{-1})^{n-1} \gamma_2^{1-n} \in \Gamma_N \quad \forall N$).

Therefore the order of the cusps around 0 is:

$$\gamma_2^0(1), \gamma_2^0(\infty), \gamma_2^{-1}(1), \gamma_2^{-1}(\infty), \dots, \gamma_2^{-N+1}(1), \gamma_2^{-N+1}(\infty) = 1, \infty, 2N-1, \frac{1}{2N-2}, 2N-3, \dots, 3, \frac{1}{2}$$

The stabilizer of ∞ is generated by $\kappa_\infty = \gamma_1^{-1}$ that turns in negative direction as $2e^{2\pi i N \lambda}$ does seen as a circle around ∞ . We get an order of cusps around ∞ :

$$1, 0, 2N-1, 2N-2, \dots, 3, 2$$

The correspondence is not unique because of symmetries. We decide to identify $0 \longleftrightarrow (0 : 1 : 1)$ and $1 \longleftrightarrow (1 : 0 : 1)$. Then the other correspondences are fixed and we get the claim. \square

6. MODULAR FORMS FOR FERMAT CURVES

The modular function and forms for $\Gamma(2)$ are modular function and forms for Γ_N as well. Based on $\lambda(z)$, $(1-\lambda)(z)$ and $\theta^2(z)$ (see Equation (4.0.3)) we can construct further modular function and forms for Γ_N .

Lemma 6.1. *Let $\lambda(z)$ and $(1-\lambda)(z)$ the modular functions introduced in Equation (4.0.3). The N -th roots*

$$x := \sqrt[N]{\lambda} \quad y := \sqrt[N]{1-\lambda}$$

exist and they are modular functions for Γ_N .

Proof: See [Ro]. \square

Lemma 6.2. *Remember the cusps a_j, b_j and c_j of Γ_N in Lemma 5.2. We have the following modular functions and forms for Γ_N with divisors as stated.*

$$\begin{aligned} \operatorname{div} \theta^2 &= \sum_{j=0}^{N-1} N b_j \\ \operatorname{div} x &= \sum_j a_j - \sum_j c_j \\ \operatorname{div} y &= \sum_j b_j - \sum_j c_j \\ \operatorname{div} (x - \zeta^j) &= N b_j - \sum_j c_j \\ \operatorname{div} (y - \zeta^j) &= N a_j - \sum_j c_j \\ \operatorname{div} (x - \epsilon \zeta^j y) &= N c_j - \sum_j c_j \end{aligned}$$

Here we have $\zeta = e^{2\pi i/N}$, $\epsilon = e^{\pi i/N}$, x as well as y are from Lemma 6.1 and $\theta^2(z)$ as in Equation (4.0.3).

Proof: See [Ro] and [Ya]. □

Now, we can construct modular forms with special zeros.

Lemma 6.3. *For $j = 0, \dots, N-1$ we define*

$$(6.3.1) \quad f_{a_j}^{\Gamma_N} := \frac{(y - \zeta^j)^N}{y^N} \theta^2$$

$$(6.3.2) \quad f_{b_j}^{\Gamma_N} := \frac{(x - \zeta^j)^N}{y^N} \theta^2$$

$$(6.3.3) \quad f_{c_j}^{\Gamma_N} := \frac{(x - \epsilon \zeta^j y)^N}{y^N} \theta^2,$$

where $\zeta = e^{2\pi i/N}$ and $\epsilon = e^{\pi i/N}$.

These all are modular forms for Γ_N of weight 2 and

$$\operatorname{div} f_{i_j}^{\Gamma_N} = N^2 i_j,$$

where $i_j \in \{a_j, b_j, c_j\}$ stands for a cusp of Γ_N (see Lemma 5.2).

Proof: Follows easily from Lemma 6.2. □

By regarding the product of q -expansions in several cusps, we recover modular forms for $\Gamma(2)$:

Lemma 6.4. *We build products for the modular forms from Lemma 6.3 under the action of the slash operator. Thereby, we get only three different results. They are for all $j \in \{0, 1, \dots, N-1\}$*

$$(6.4.1) \quad \prod_{\gamma \in \Gamma_N \setminus \Gamma(2)} f_{a_j}^{\Gamma_N} |_2 \gamma(z) = (-1)^{N^2} \theta^{2N^2}(z) \left(\frac{\lambda(z)}{1 - \lambda(z)} \right)^{N^2}$$

$$(6.4.2) \quad \prod_{\gamma \in \Gamma_N \setminus \Gamma(2)} f_{b_j}^{\Gamma_N} |_2 \gamma(z) = (-1)^{N^2} \theta^{2N^2}(z)$$

$$(6.4.3) \quad \prod_{\gamma \in \Gamma_N \setminus \Gamma(2)} f_{c_j}^{\Gamma_N} |_2 \gamma(z) = \theta^{2N^2}(z) \left(\frac{1}{1 - \lambda(z)} \right)^{N^2}.$$

Proof: We have to apply all matrices from (5.5.1) to the f_{i_j} .

The transformational behavior of the form θ^2 is known since $\theta^2 \in M_2(\Gamma(2))$. The behavior of x and y is (according to [Ya])

$$\begin{aligned} x|_0\gamma_1 &= \zeta^{-1}x & x|_0\gamma_2 &= \zeta^{-1}x \\ y|_0\gamma_1 &= \zeta^{-1}y & y|_0\gamma_2 &= y. \end{aligned}$$

With this information direct calculations yield the claim. \square

7. KRONECKER LIMIT FORMULAS FOR FERMAT CURVES

We want to establish Kronecker limit formulas for the group Γ_N . From Proposition 3.4 we know that the modular forms introduced in Lemma 6.3 are suitable. Missing is the constant A that occurs in Proposition 3.4.

To calculate that constant, we will use the following trick: The constant A can be calculated by comparing Fourier expansions. Since the constant is independent of the cusp in which the Fourier expansion is taken, we may add several expansions and get a multiple of A . By such a procedure we can obtain A because a suitable sum of expansions leads to known formulas.

Lemma 7.1. *Let $\Gamma_N \subset \Gamma(2)$ be the subgroup associated to the N -th Fermat curve, S_j a cusp of Γ_N and $f_j^{\Gamma_N}$ the modular form for the cusp S_j according to Lemma 6.3. It holds:*

$$(7.1.1) \quad \sum_{\gamma \in \Gamma_N \setminus \Gamma(2)} 4\pi \lim_{s \rightarrow 1} \left(E_j^{\Gamma_N}(\gamma z, s) - \frac{1}{\text{vol}(\Gamma_N)(s-1)} \right) = \\ - \frac{1}{N^2} \sum_{\gamma \in \Gamma_N \setminus \Gamma(2)} \log \|f_j^{\Gamma_N}|_0 \gamma(z)\|^2 + 4 \left(\frac{\zeta'(-1)}{\zeta(-1)} - \log(4\pi) + 1 + \frac{1}{6} \log(2) - \frac{1}{2} \log(N) \right)$$

Proof: The left hand side of Equation (7.1.1) had been calculated in Corollary 2.12:

$$4\pi \lim_{s \rightarrow 1} \left(E_j^{\Gamma(2)}(z, s) - \frac{1}{\text{vol}(\Gamma(2))(s-1)} \right) - 2 \log(N)$$

For the right hand side we realize that

$$\sum_{\gamma \in \Gamma_N \setminus \Gamma(2)} \log \|f_j^{\Gamma_N}|_0 \gamma(z)\|^2 = \log \left\| \prod_{\gamma \in \Gamma_N \setminus \Gamma(2)} f_j^{\Gamma_N}|_0 \gamma(z) \right\|^2.$$

Then we use Lemma 6.4 to see that the sum yields $N^2 \log \|G_j(z)\|^2$ (the form G_j is one from equations (4.0.4) to (4.0.6)). Therefore we get the claim by comparing the formulas here with the Kronecker limit formula for $\Gamma(2)$ (Equation (4.1.1)). \square

From the sum formula (7.1.1) we derive individual Kronecker limit formulas.

Theorem 7.2 (Kronecker limit formula for Γ_N). *Let S_j be a cusp of Γ_N , the subgroup associated to the N -th Fermat curve (see Lemma 5.3), and let $f_j^{\Gamma_N} \in M_2(\Gamma_N)$ be the corresponding modular form defined in Lemma 6.3. We have*

$$(7.2.1) \quad 4\pi \lim_{s \rightarrow 1} \left(E_j^{\Gamma_N}(z, s) - \frac{1}{\text{vol}(\Gamma_N)(s-1)} \right) = \\ - \frac{1}{N^2} \log \|f_j^{\Gamma_N}(z)\|^2 + \frac{4}{N^2} \left(\frac{\zeta'(-1)}{\zeta(-1)} - \log(4\pi) + 1 + \frac{1}{6} \log(2) - \frac{1}{2} \log(N) \right).$$

Proof: From Proposition 3.4 follows, that there is an identity

$$4\pi \lim_{s \rightarrow 1} \left(E_j^{\Gamma_N}(z, s) - \frac{1}{\text{vol}(\Gamma_N)(s-1)} \right) = - \frac{1}{N^2} \log \|f_j^{\Gamma_N}(z)\|^2 + A,$$

where A is a constant. Then Remark 3.5 explains that A is $\frac{1}{N^2}$ times the constant that occurs in Lemma 7.1. \square

8. SCATTERING CONSTANTS

If we take the Fourier expansions in Equation 7.2.1 and compare coefficients, we can get the scattering constants for Γ_N . We only know the Fourier expansion for $f_j^{\Gamma_N}$ in the cusp ∞ . But because of symmetries of the Fermat curve, this is sufficient to get all scattering constants.

Theorem 8.1. *The scattering constants for Γ_N , the subgroup associated to the N -th Fermat curve, are:*

If both cusps are the same, then

$$(8.1.1) \quad C_{jj}^{\Gamma_N} = \frac{1}{6N^2} \left(C^{\Gamma(1)} - \frac{1}{\pi} ((12N+2)\log(2) + (-3N+6)\log(N)) \right).$$

If $S_j \neq S_k$ and $\beta_N(S_j) \neq \beta_N(S_k)$, then

$$(8.1.2) \quad C_{kj}^{\Gamma_N} = \frac{1}{6N^2} \left(C^{\Gamma(1)} - \frac{1}{\pi} (2\log(2) + 6\log(N)) \right).$$

If $S_j \neq S_l$ but $\beta_N(S_j) = \beta_N(S_l)$, then

$$(8.1.3) \quad C_{lj}^{\Gamma_N} = \frac{1}{6N^2} \left(C^{\Gamma(1)} - \frac{1}{\pi} \left(2\log(2) + 6\log(N) + 3N \log |1 - \zeta_N^{l-j}| \right) \right).$$

The number $\zeta_N^{l-j} = \zeta_N^l (\zeta_N^j)^{-1}$ is given by the N -th roots of unity which determine the cusps S_l and S_j in Lemma 5.2.

Proof: In the case that the second cusp $S_j = \infty$ we get the scattering constants via comparison of the first coefficients in the realizations of Equation (7.2.1). We expand both sides of (7.2.1) in ∞ and calculate the constant term:

The constant term of the Eisenstein series at $s = 1$ is (see Equation (2.9.1))

$$\delta_{jk} \frac{2\pi y}{N} + 4\pi \tilde{C}_{jk}^{\Gamma_N} - \frac{2}{N^2} \log(y).$$

On the other side of Equation (7.2.1) we need the constant term of the q -expansion of $f_j^{\Gamma_N}$. We get the constant term if we look at the q -expansions of θ^2 , λ as well as $1 - \lambda$ and calculate their roots. The result is

$$\text{constant term } \left(\log \|f_j^{\Gamma_N}\|^2 \right) = \begin{cases} 2\log(y) - 2N\pi y + N(\log(16) - \log(N)) & \text{if } S_j = c_0 \\ 2\log(y) + \log |1 - \zeta_N^j| & \text{if } S_j = c_j \\ 2\log(y) & \text{if } S_j \in \{a_j, b_j\}, \end{cases}$$

where the names of the cusps come from Equation (5.2.2). The root ζ^j in the second case is determined by the cusp $c_j = (\epsilon \zeta_N^j : 1 : 0)$ ($j \in \{1, \dots, N-1\}$).

By taking the constant from Theorem 7.2 and the difference $\tilde{C}_{jk}^{\Gamma_N} = C_{jk}^{\Gamma_N} + \frac{1}{2N^2\pi} \log(2N)$ into account, we derive all formulas from the statement.

Now, all that remains to be shown is that these formulas generalize to arbitrary second cusp.

Via the definition of Eisenstein series it is easy to show that for $\gamma \in \Gamma(1)$

$$E_{\gamma^{-1}(j)}^{\Gamma}(\gamma_l^{-1}z, s) = E_j^{\gamma^{-1}\Gamma\gamma}(\gamma_l^{-1}\gamma^{-1}z, s)$$

holds. Since $\Gamma_N \triangleleft \Gamma(1)$, we have $\Gamma_N = \gamma^{-1}\Gamma_N\gamma$ and derive

$$C_{jl}^{\Gamma_N} = C_{\gamma(j), \gamma(l)}^{\Gamma_N}.$$

Therefore, to get all scattering constants we have to find elements of $\Gamma(1)$ that map ∞ to all cusps and to find out to where these matrices move the second cusp.

The conjugation with $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (to get 0) or $TS = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (to get 1) give (for even j, l and odd k in the range $1, 2, \dots, 2N$)

$$\begin{aligned} C_{1/l, \infty}^{\Gamma_N} &= C_{2N-l, 0}^{\Gamma_N} & C_{1/l, \infty}^{\Gamma_N} &= C_{2N-l+1, 1}^{\Gamma_N} \\ C_{j, \infty}^{\Gamma_N} &= C_{1/(2N-j), 0}^{\Gamma_N} & C_{j, \infty}^{\Gamma_N} &= C_{1/j, 1}^{\Gamma_N} \\ C_{k, \infty}^{\Gamma_N} &= C_{2N-k, 0}^{\Gamma_N} & C_{k, \infty}^{\Gamma_N} &= C_{2N-k+1, 1}^{\Gamma_N}. \end{aligned}$$

(To see this we use

$$\begin{aligned} -l \sim_{\Gamma_N} 2N-l & \quad \text{via} \quad \gamma_1^N \\ -1/j \sim_{\Gamma_N} 1/(2N-j) & \quad \text{via} \quad \gamma_2^N \\ -1/k \sim_{\Gamma_N} 2N-k & \quad \text{via} \quad \gamma_1^{(2N+k-1)/2} (\gamma_2 \gamma_1^{-1})^{(k-1)/2} \gamma_2^{(1-k)/2} \\ (j-1)/j \sim_{\Gamma_N} 1/j & \quad \text{via} \quad \gamma_2^{j/2} \gamma_1^{(j-2N)/2} (\gamma_1 \gamma_2^{-1})^{j/2} \\ (k-1)/k \sim_{\Gamma_N} 2N-k+1 & \quad \text{via} \quad \gamma_1^{(2N-k+1)/2} \gamma_2^{(k-1)/2} (\gamma_2 \gamma_1^{-1})^{(1-k)/2} \end{aligned}$$

to identify the cusps.)

Hence, all scattering constants for 0, 1 or ∞ as second cusp are known.

With γ_1 in the cases 0 as well as 1 and γ_2 in the case of ∞ we generalize to all cusps. (In the only cases we really need, i.e. the ones where both cusps involved are equivalent under $\Gamma(2)$, there occur no further difficulties when we try to identify the resulting cusps in the system of representatives.)

Thereby, we will find the formulas from the statement when we use the correspondence of cusps in Proposition 5.6 and realize that $|1 - \zeta_N^j| = |1 - \zeta_N^{N-j}|$. \square

Remark 8.2. There is an alternative method to determine the scattering constants for the Fermat curves by means of Arakelov theory. U. Kühn [Kü] showed how scattering constants give arithmetic intersection numbers in the infinite places. Together with the intersection numbers in the finite places that Ch. Curilla [Cu] calculated in his PhD-thesis, we could get the scattering constants without using Kronecker limit formulas and the symmetries of the Fermat curves by studying arithmetic intersection numbers in the cusps.

Remark 8.3. We can approximate scattering constants for the Fermat curves numerically. The means for that has been developed by the author in her Diplomarbeit [Po1] and her Dissertation [Po2]. For some small N , the structure of the scattering matrix, i.e. the symmetries, and the values of the scattering constants were replicated numerically.

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